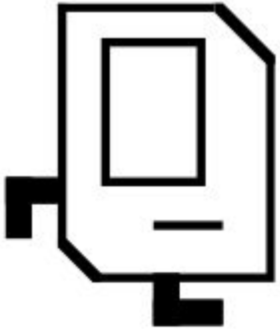
First-Order Logic

Part Two

CS198 Section Leading



cs198@cs.stanford.edu

Who should section lead?

For this round of applications, we are looking for applicants have completed the equivalent of CS106B… and that's you!

We are looking for section leaders from all backgrounds who can relate to students and clearly explain concepts.

What do section leaders do?

- Teach a weekly 50 minute section
- Help students in the LaIR
- Grade CS106 assignments
- Hold IGs with students
- Grade midterms and finals
- Get paid $18.50/hour (more with seniority)
- Have fun!

Time and requirements

You'll need to:

- Section lead for **two quarters!**
- Take CS198 for 3-4 units (1st quarter only)
- Attend staff meetings (Monday, 4:30-5:30PM)
- Attend Monday workshops (7:30-9pm) for first 4 weeks of first quarter
- Attend Wednesday workshops (based on availability) for first 4 weeks of first quarter
- Fulfill all teaching, LaIR, and grading responsibilities

Why section lead?

- “Learn to teach; teach to learn”
- Work directly with students
- Participate in fun events
- Join an amazing group of people
- Leave your mark on campus

Participate in fun events






- LaIR Formal
- Special D
- Movie Nights
- BAWK
- Lecturer Hangouts
- New SL Picnic
- Swag
- And more!

Apply Now

Application is open now!

Deadline: **Thursday, February 1st at 11:59PM PT**

Online application: cs198.stanford.edu

Contact us: cs198@cs.stanford.edu

Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about many objects at once.

Some spider is radioactive.

$$\exists s.\ (Spider(s) \land Radioactive(s))$$

∃ is the ***existential quantifier*** and says "for some choice of s, the following is true."

“For any natural number n,
n is even if and only if n^2 is even”

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

$\forall$ is the ***universal quantifier*** and says “for any choice of n, the following is true.”

“Some *P* is a *Q*”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \land Q(x))$$

If x is an example, it must have property P on top of property Q.

“All P's are Q's”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If x is a counterexample, it *must* have property P but not have property Q.

New Stuff!

The Aristotelian Forms

"All *A*s are *B*s"

$\forall x.\ (A(x) \to B(x))$

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

"No *A*s are *B*s"

$\forall x.\ (A(x) \to \neg B(x))$

"Some *A*s aren't *B*s"

$\exists x.\ (A(x) \land \neg B(x))$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

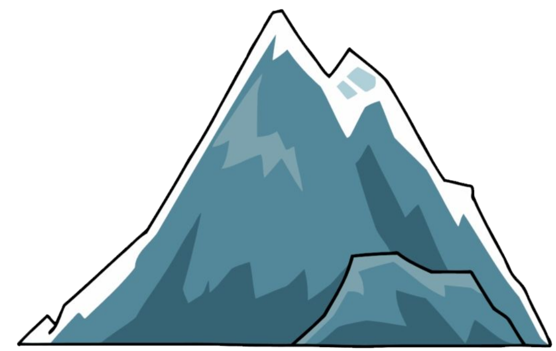
- *Person*(*p*), which states that *p* is a person, and
- *Loves*(*x*, *y*), which states that *x* loves *y*,

write a sentence in first-order logic that means “every person loves someone else.”

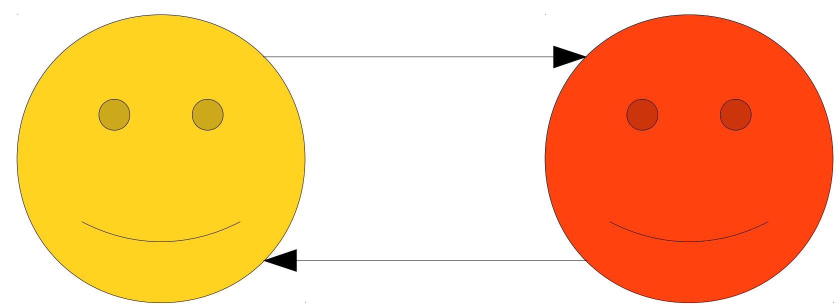
$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Things to Watch Out For

$$\forall p.\ (Person(p) \land \\ \quad \exists q.\ (Person(q) \land p \neq q \land \\ \qquad Loves(p, q) \\ \quad) \\ )$$



Do the "Mt. Everest Test" – plug in something we don't care about (eg. Mt. Everest)



Things to Watch Out For

⚠ ~~$\forall p.\ (Person(p)\ \land$~~

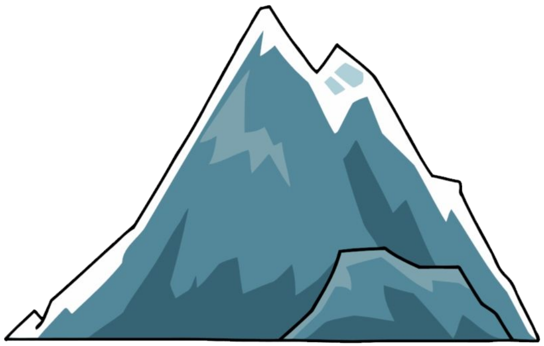
~~$\exists q.\ (Person(q) \land p \neq q\ \land$~~

~~$Loves(p, q)$~~

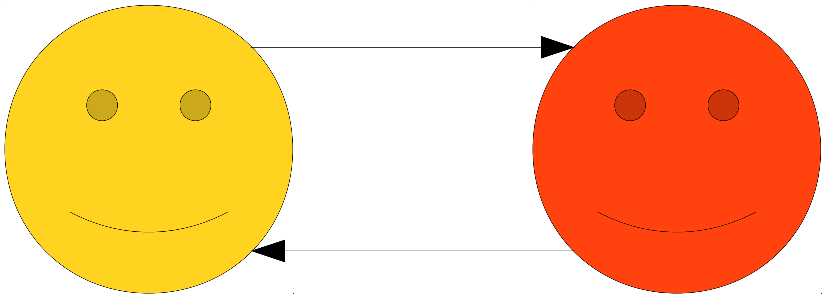
~~$)$~~

~~$)$~~

⚠



We don't want to pair $\forall$ with $\land$ because we don't want irrelevant choices of p to accidentally make our statement false.



Time-Out for Announcements!

Problem Set One Solutions

- We've just posted solutions to Problem Set One. They're linked from the main PS1 page.
- We recommend you read over our solution set before finishing PS2.
  - You'll get to see examples of polished written proofs.
  - Each problem has a "Why We Asked This Question" section, which gives some context.
  - We may have solved the problem differently than you, and this will give you more perspectives to use.
- We'll aim to have PS1 graded and returned by Wednesday.

Back to CS103!

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

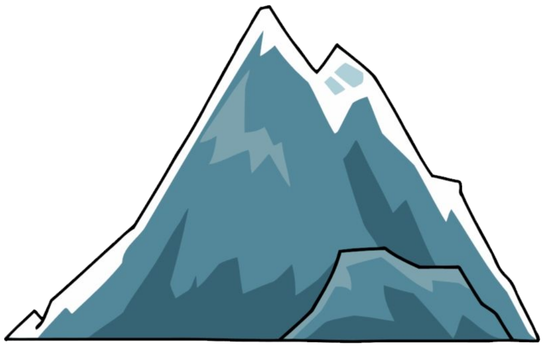
write a sentence in first-order logic that means “there is a person that everyone else loves.”

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

Things to Watch Out For

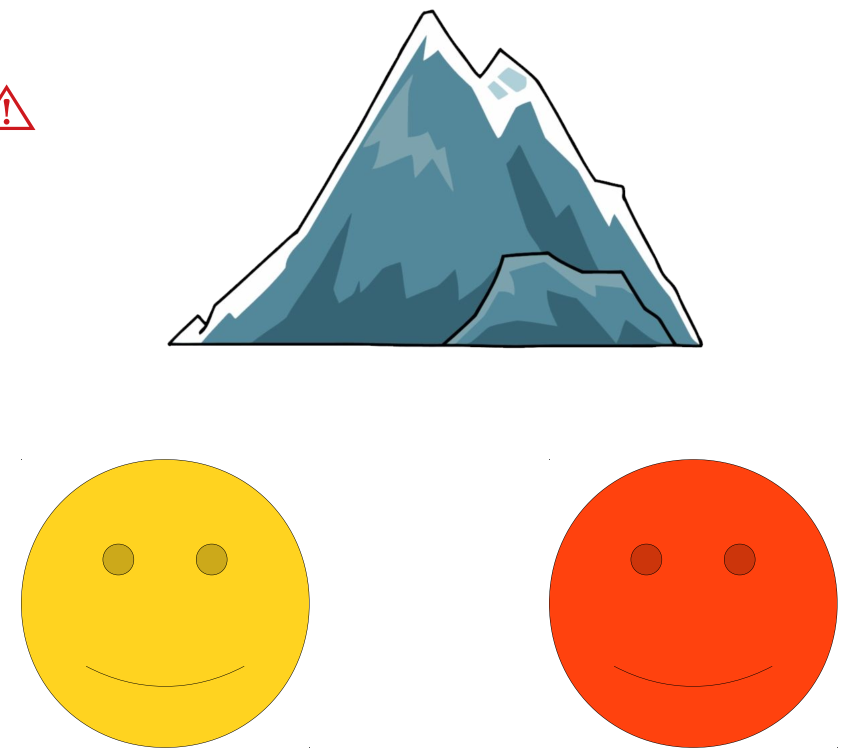
⚠ $\exists p.\ \forall q.\ ((Person(p) \land Person(q) \land p \neq q) \to Loves(q, p))$

⚠



Things to Watch Out For

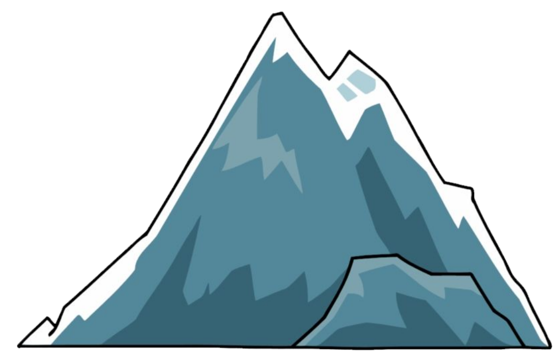
⚠ $\exists p.\ \forall q.\ ($ ~~$(Person(p) \land Person(q) \land p \neq q)$~~ $\to Loves(q, p)$
$)$ ⚠

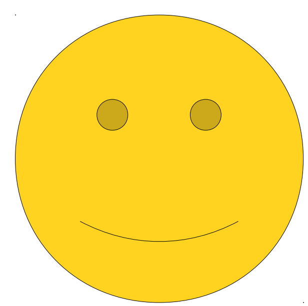


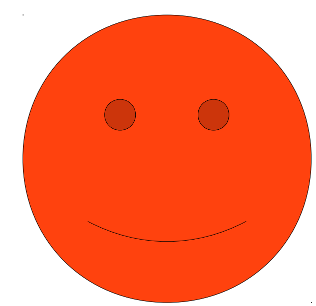
Since the antecedent is false, this entire implication evaluates to true, even though there's no person that everyone loves in this world.

Things to Watch Out For

⚠ $\exists p.\ \forall q.\ ($~~$(Person(p) \land Person(q) \land p \neq q)$~~ $\to$
$Loves(q, p)$
$)$ ⚠



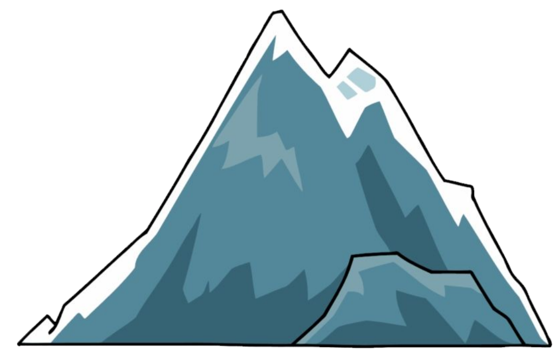




Don't put all of your quantifiers up front – it often changes the meaning of the statement!

Things to Watch Out For

⚠ $\exists p.$ $\forall q.$ (~~$(Person(p) \land Person(q) \land p \neq q)$~~ $\to$
$Loves(q, p)$
) ⚠



Also, don't pair ∃ with →
because we don't want
irrelevant choices of *p* to
accidentally make our
statement true.

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: “Every person loves someone else”

For every person… $\forall p.\ (Person(p) \to$

… there is another person $\exists q.\ (Person(q) \land p \neq q \land$

… … they love $Loves(p, q)$

$)$

$)$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "There is someone everyone else loves."

There is a person…

… that everyone else …

… loves.

$$
\begin{aligned}
&\exists p.\ (Person(p)\ \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \rightarrow \\
&\qquad Loves(q, p) \\
&\quad) \\
&)
\end{aligned}
$$

For Comparison

For every person… $\forall p.\ (Person(p) \to$

… there is another person $\exists q.\ (Person(q) \land p \neq q \land$

…

… they love $Loves(p, q)$

$)$

$)$

There is a person… $\exists p.\ (Person(p) \land$

… that everyone else … $\forall q.\ (Person(q) \land p \neq q \to$

… loves. $Loves(q, p)$

$)$

$)$

Restricted Quantifiers

Quantifying Over Sets

- The notation

  $$\forall x \in S.\ P(x)$$

  means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

- The notation

  $$\exists x \in S.\ P(x)$$

  means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ P(x)$$

$$\exists x \in S.\ P(x)$$

  is allowed for quantifying over sets.
- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x$ with $P(x)$. $Q(x)$ ⚠

⚠ $\forall y$ such that $P(y) \land Q(y)$. $R(y)$. ⚠

⚠ $\exists P(x)$. $Q(x)$ ⚠

Quantifier Ordering

Quantifier Ordering

- As you saw in the two translations we did today, the order of quantifiers can dramatically change the meaning of a statement. Why is that?

Quantifier Ordering

- Consider the following example:
  - $\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ m < n$
  - $\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ m < n$

This first statement says "every natural number *m* has a larger natural number *n*". This is true: if you give me a number *m*, I can pick *n* to be *m* + 1.

The second statement says "there is a natural number *n* that's greater than every natural number *m*". This is false: there's no largest natural number!

Quantifier Ordering

- Consider the following example:
  - $\forall m \in \mathbb{N}.\ \exists n \in \mathbb{N}.\ m < n$
  - $\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ m < n$
- In the first statement, we're allowed to pick a different *n* for each *m*.
- In the second statement, the same *n* needs to work for all choices of *m*.

Quantifier Ordering

- The statement

  $$\forall x.\ \exists y.\ P(x, y)$$

  means "for any choice of x, there's some choice of y where $P(x, y)$ is true."
- The choice of y can be different every time and can depend on x.

Quantifier Ordering

- The statement

$$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some *x* where for any choice of *y*, we get that *P*(*x*, *y*) is true."
- Since the inner part has to work for any choice of *y*, this places a lot of constraints on what *x* can be.

Order matters when mixing existential and universal quantifiers!

Mechanics: Negating Statements

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	There is an x where $P(x)$ is false.
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	For all objects x, $P(x)$ is false.
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	There is an x where $P(x)$ is true.
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	For all objects x, $P(x)$ is true.

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For all objects x, $P(x)$ is true.	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	There is an x where $P(x)$ is true.	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For all objects x, $P(x)$ is false.	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	There is an x where $P(x)$ is false.	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

$$\neg\forall x.\ A \quad \textit{is equivalent to} \quad \exists x.\ \neg A$$

$$\neg\exists x.\ A \quad \textit{is equivalent to} \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$
$$\exists x.\ \neg\exists y.\ Loves(x, y)$$
$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \quad \textit{is equivalent to} \quad p \to \neg q$$

$$\neg(p \to q) \quad \textit{is equivalent to} \quad p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says “there is a cute puppy”?

$$\boldsymbol{\exists x. \ (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg\exists x. \ (Puppy(x) \land Cute(x))$$

$$\forall x. \ \neg(Puppy(x) \land Cute(x))$$

$$\forall x. \ (Puppy(x) \to \neg Cute(x))$$

- This says “no puppy is cute.”
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

Expressing Uniqueness

Using the predicate

- $WayToFindOut(w)$, which states that w is a way to find out,

write a sentence in first-order logic that means “there is only one way to find out.”

$$
\begin{aligned}
&\exists w.\ (\mathit{WayToFindOut}(w)\ \land \\
&\quad \forall x.\ (\mathit{WayToFindOut}(x) \rightarrow x = w) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special “uniqueness quantifier” used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

Next Time

- ***Functions***
  - How do we model transformations and pairings?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?